Advance Higher Unit 1 — Homework 5 — Solutions

1.(a) $\left(\begin{array}{ccc|c} 2 & 0 & 1 & 2 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & 3 \end{array}\right)$

$r_1 \leftrightarrow r_2$ $\left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 2 & 0 & 1 & 2 \\ 0 & 1 & -2 & 3 \end{array}\right)$

new $r_2 = r_2 - 2r_1$ $\left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & -2 & -1 & 4 \\ 0 & 1 & -2 & 3 \end{array}\right)$

new $r_3 = 2r_3 + r_2$ $\left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & -2 & -1 & 4 \\ 0 & 0 & -5 & 10 \end{array}\right)$

$\Rightarrow -5z = 10$
$z = -2$

$-2y - z = 4$
$-2y = 2$
$y = -1$

$x + y + z = -1$
$x = -1 + 1 + 2$
$= 2$

$x = 2, \; y = -1, \; z = -2.$

(b) $\left(\begin{array}{ccc|c} 2 & 1 & 1 & 4 \\ 2 & -1 & -2 & 1 \\ 1 & 0 & -1 & 2 \end{array}\right)$

$r_1 \leftrightarrow r_3$ $\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 2 & -1 & -2 & 1 \\ 2 & 1 & 1 & 4 \end{array}\right)$

1(b) continued

new $r_2 = r_2 - 2r_1$

new $r_3 = r_3 - r_2$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & -1 & 0 & -3 \\ 0 & 2 & 3 & 3 \end{array}\right)$$

new $r_3 = r_3 + 2r_2$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 3 & -3 \end{array}\right)$$

$3z = -3$ $\quad$ $-y = -3$ $\quad$ $x - z = 2$

$z = -1$ $\quad$ $y = 3$ $\quad$ $x = 2 + z$

$x = 1$

$\underline{\underline{x = 1, \ y = 3, \ z = -1.}}$

(c) $$\left(\begin{array}{ccc|c} 3 & -1 & -1 & -11 \\ 1 & -1 & 1 & -9 \\ 1 & 2 & -2 & 9 \end{array}\right)$$

$r_1 \longleftrightarrow r_3$ $$\left(\begin{array}{cccc} 1 & 2 & -2 & 9 \\ 1 & -1 & 1 & -9 \\ 3 & -1 & -1 & -11 \end{array}\right)$$

new $r_2 = r_1 - r_2$

new $r_3 = r_3 - 3r_2$

$$\left(\begin{array}{ccc|c} 1 & 2 & -2 & 9 \\ 0 & 3 & -3 & 18 \\ 0 & 2 & -4 & 16 \end{array}\right)$$

1(c) continued.

new $r_3 = 3r_3 - 2r_2$

$$\left(\begin{array}{ccc|c} 1 & 2 & -2 & 9 \\ 0 & 3 & -3 & 18 \\ 0 & 0 & -6 & 12 \end{array}\right)$$

$\Rightarrow \quad -6z = 12 \qquad 3y - 3z = 18 \qquad x + 2y - 2z = 9$

$z = -2 \qquad y - z = 6 \qquad x = 9 + 2z - 2y$

$y = 4 \qquad x = -3.$

$\underline{\underline{x = -3, \; y = 4, \; z = -2.}}$

2. $y = ax^2 + bx + c.$

$(1,2) \Rightarrow 2 = a + b + c$

$(2,5) \Rightarrow 5 = 4a + 2b + c$

$(-1,8) \Rightarrow 8 = a - b + c$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 4 & 2 & 1 & 5 \\ 1 & -1 & 1 & 8 \end{array}\right)$$

new $r_2 = r_2 - 4r_1$

new $r_3 = r_3 - r_1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & -3 & -3 \\ 0 & -2 & 0 & 6 \end{array}\right)$$

$r_2 \leftrightarrow r_3$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & 6 \\ 0 & -2 & -3 & -3 \end{array}\right)$$

2, continued.

new $r_3 = r_3 - r_2$ $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & 6 \\ 0 & 0 & -3 & -9 \end{array}\right)$

$-3c = -9$ $\quad$ $-2b = 6$ $\quad$ $a + b + c = 2$

$c = 3$ $\quad$ $b = -3$ $\quad$ $a = 2 - b - c$

$a = 2$

Equation of parabola $\underline{\underline{y = 2x^2 - 3x + 3.}}$

3. $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 60 \\ 2 & 3 & 1 & 85 \\ 3 & 1 & p & 105 \end{array}\right)$

new $r_2 \rightarrow r_2 - 2r_1$
new $r_3 \rightarrow r_3 - 3r_1$ $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 60 \\ 0 & -1 & -1 & -35 \\ 0 & -5 & p-3 & -75 \end{array}\right)$

new $r_3 \rightarrow r_3 - 5r_2$ $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 60 \\ 0 & -1 & -1 & -35 \\ 0 & 0 & p+2 & 100 \end{array}\right)$

$(p + 2)z = 100$

* For no solution $p + 2 = 0 \Rightarrow \underline{\underline{p = -2}}$. Inconsistency.

4. $2x + y - 3z = 5$
$x - 2y + 3z = 1$
$2x - y + az = b.$

$$\left(\begin{array}{ccc|c} 2 & 1 & -3 & 5 \\ 1 & -2 & 3 & 1 \\ 2 & -1 & a & b \end{array}\right)$$

$r_2 \leftrightarrow r_1$
$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 2 & 1 & -3 & 5 \\ 2 & -1 & a & b \end{array}\right)$$

new $r_2 = r_2 - 2r_1$
new $r_3 = r_3 - r_2$
$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -9 & 3 \\ 0 & -2 & a+3 & b-5 \end{array}\right)$$

new $r_3 = 3r_3 - 2r_2$
$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -9 & 3 \\ 0 & 0 & 3a+27 & 3b-21 \end{array}\right)$$

(a) For no solution $3a + 27 = 0$
$a = -9$
and $3b - 21 \neq 0$
$b \neq 7$
$\underline{\underline{a = -9 \quad b \neq 7}}$

(b) For infinitely many solutions $\underline{\underline{a = -9, \ b = 7.}}$

5. <u>"Ill-conditioned"</u>

Ill conditioning occurs when a small change in any of the values in a system of equations leads to a disproportionate change in the solutions.

(a) $7x + 5y = 19$

$4x + 3y = 11$ Let $4x = 11 - 3y$

$$x = -\frac{3}{4}y + \frac{11}{4}$$

$$7x + 5y = 19$$

$$7\left(-\frac{3}{4}y + \frac{11}{4}\right) + 5y = 19$$

$$-\frac{21}{4}y + \frac{77}{4} + 5y = 19$$

$$-\frac{21y}{4} + \frac{77}{4} + \frac{20y}{4} = \frac{76}{4}$$

$$-y = -1$$

$$y = 1 \qquad x = -\frac{3}{4} + \frac{11}{4}$$

$$x = 2$$

$x = 2$, $y = 1$.

Test for ill-conditioning.

When $7x + 5y = 18.5$

$4x + 3y = 10.5$ Solution $x = -2\frac{1}{4}$, $y = -\frac{1}{2}$

When $7x + 5y = 18.5$

$4x + 3y = 11$ Solution $x = -\frac{1}{2}$, $y = 3$

When $7x + 5y = 19$

$4x + 3y = 10.5$ Solution $x = 4\frac{7}{8}$, $y = -2\frac{1}{2}$

[illegible] ill-conditioned