HIGHER MATHS 2009 — PAPER 2 SOLUTIONS

1. $y = x^3 - 3x^2 - 9x + 12$

$\frac{dy}{dx} = 3x^2 - 6x - 9$

for stat pts $\frac{dy}{dx} = 0$

$\Rightarrow \quad 3x^2 - 6x - 9 = 0$

$x^2 - 2x - 3 = 0$

$(x+1)(x-3) = 0$

$x = -1, 3$

$\Rightarrow$ Stat pts at $(-1, 17)$ & $(3, -15)$

when $x = -1$,

$y = (-1)^3 - 3(-1)^2 - 9(-1) + 12$

$y = 17$

when $x = 3$

$y = (3)^3 - 3(3)^2 - 9(3) + 12$

$y = -15$

x	$\rightarrow$	-1	$\rightarrow$	3	$\rightarrow$
$\frac{dy}{dx}$	$+$	0	$-$	0	$+$
shape	/	—	\	—	/

when $x = -2$

$\frac{dy}{dx} = 3(-2)^2 - 6(-2) - 9$

$= 15$

when $x = 0$

$\frac{dy}{dx} = 3(0)^2 - 6(0) - 9$

$= -9$

when $x = 4$

$\frac{dy}{dx} = 3(4)^2 - 6(4) - 9$

$= 15$

Max TP at $(-1, 17)$

Min TP at $(3, -15)$

2. (a) $f(x) = 3x + 1 \qquad g(x) = x^2 - 2$

(i)
$$\begin{aligned} p(x) &= f(g(x)) \\ &= f(x^2-2) \\ &= 3(x^2-2)+1 \\ p(x) &= 3x^2 - 5 \end{aligned}$$

(ii)
$$\begin{aligned} q(x) &= g(f(x)) \\ &= g(3x+1) \\ &= (3x+1)^2 - 2 \\ &= 9x^2 + 6x + 1 - 2 \\ q(x) &= 9x^2 + 6x - 1 \end{aligned}$$

(b)
$$\begin{aligned} p'(x) &= q'(x) \\ 6x &= 18x + 6 \\ -6 &= 12x \\ x &= -\tfrac{1}{2} \end{aligned}$$

3. (a) (i) $x = 1$

1	8	11	-20
	1	9	20
1	9	20	0

$\Rightarrow x = 1$ is a root

(ii) Quotient $= x^2 + 9x + 20$

$$\begin{aligned} \Rightarrow \quad & x^3 + 8x^2 + 11x - 20 \\ &= (x-1)(x^2+9x+20) \\ &= (x-1)(x+4)(x+5) \end{aligned}$$

(b)
$$\begin{aligned} \log_2(x+3) + \log_2(x^2+5x-4) &= 3 \\ \log_2[(x+3)(x^2+5x-4)] &= 3 \\ (x+3)(x^2+5x-4) &= 2^3 \\ x^3 + 5x^2 - 4x + 3x^2 + 15x - 12 &= 8 \\ x^3 + 8x^2 + 11x - 20 &= 0 \\ (x-1)(x+4)(x+5) &= 0 \\ x &= -5, -4, 1 \\ \Rightarrow x &= 1 \quad * \end{aligned}$$

* $(x+3)$ & (x^2+5x-4) must be greater than zero

4. (a) $P(5,10)$ $\quad (x+1)^2 + (y-2)^2 = 100$

$$(5+1)^2 + (10-2)^2 = 100$$

$$36 + 64 = 100$$

$$LHS \equiv RHS$$

$\Rightarrow P(5,10)$ lies on the circle

(b) $C(-1,2)$

$$m_r = \frac{10-2}{5-(-1)}$$

$$m_r = \frac{8}{6}$$

$$m_r = \frac{4}{3}$$

$\Rightarrow m_T = -\frac{3}{4}$ as $m_1 m_2 = -1$ for $\perp$ lines

$$\vec{PC} = \begin{pmatrix}-6\\-8\end{pmatrix}$$

$$\Rightarrow \vec{CQ} = \begin{pmatrix}-6\\-8\end{pmatrix}$$

$$\Rightarrow Q(-7,-6)$$

$$y-(-6) = -\frac{3}{4}(x-(-7))$$

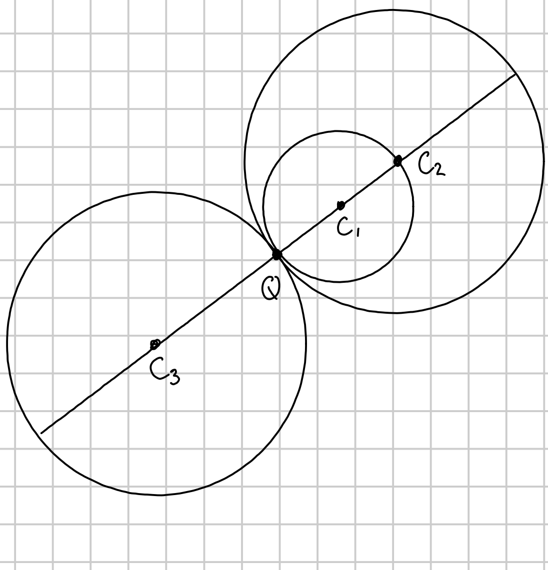
$$y+6 = -\frac{3}{4}(x+7)$$

$$4y+24 = -3x-21$$

$$4y = -3x-45$$

equation of tangent at Q is $3x + 4y + 45 = 0$

(c)



$C_2(5,10)$ $\quad r = 20$

$$C_2: (x-5)^2 + (y-10)^2 = 400$$

$$\vec{C_2Q} = \vec{QC_3} = \begin{pmatrix}-12\\-16\end{pmatrix}$$

$$\Rightarrow C_3(-19,-22)$$

$$C_3: (x+19)^2 + (y+22)^2 = 400$$

5. (a) $g(x) = 3\cos 2x$

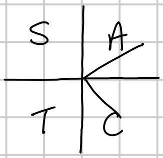
$\Rightarrow \quad m = 3, \quad n = 2$

(b) $3\cos 2x = -4\cos 2x + 3$

* Note the domain is in radians you should set your calculator to radian mode.

$7\cos 2x = 3$

$\cos 2x = \frac{3}{7}$



$2x = \cos^{-1}\left(\frac{3}{7}\right)$

$2x = 1.1, 5.2$ *

$2\pi - 1.1 = 5.2$

$(360^\circ - 64.6^\circ = 295.4^\circ)$

$2x^\circ = \cos^{-1}\left(\frac{3}{7}\right)$

$2x^\circ = 64.6^\circ, 295.4^\circ$

$2x = 64.6 \times \frac{\pi}{180}, \quad 295.4 \times \frac{\pi}{180}$

$2x = 1.1, 5.2$

** If you continue in degrees you must convert your solutions to radians at the end

continued: $2x = 1.1, 5.2$ radians

$x = 0.6, 2.6$ radians

(further solutions not required as $0 \leqslant x \leqslant \pi$)

$g(0.6) = 3\cos 2(0.6)$ *

$= 1.3$

$g(2.6) = 3\cos 2(2.6)$ *

$= 1.3$

* using rounded x-values produces $y = 1.1$ \& 1.4 use unrounded values for more precise y-values

Points of intersection at $(0.6, 1.3)$ and $(2.6, 1.3)$

(b) $A = \int_{0.6}^{2.6} \left(-4\cos 2x + 3 - 3\cos 2x\right) dx$

$= \int_{0.6}^{2.6} \left(3 - 7\cos 2x\right) dx$

$= \left[3x - \frac{7}{2}\sin 2x\right]_{0.6}^{2.6}$

$= \left(3(2.6) - \frac{7}{2}\sin 2(2.6)\right) - \left(3(0.6) - \frac{7}{2}\sin 2(0.6)\right)$

$= 12.4$

6. (a) $\ast$ 1.6% increase = 0.016 $\ast$

$N = 61e^{(0.016\times 14)}$

$N = 76.3$ million

(b) $10.2 = 5.1e^{0.0043t}$ $\ast$ 0.43% = 0.0043 $\ast$

$2 = e^{0.0043t}$

$\log_e 2 = \log_e e^{0.0043t}$

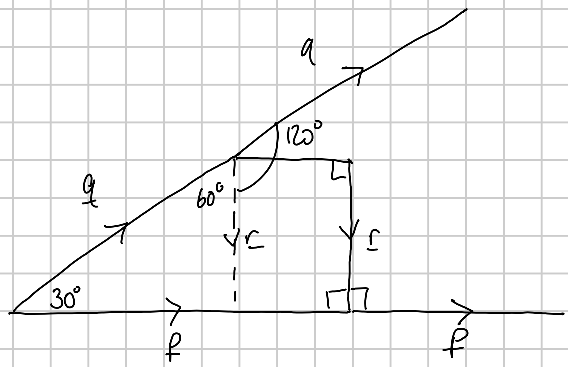
$\log_e 2 = 0.0043t \log_e e$

$\log_e 2 = 0.0043t$

$t = \dfrac{\log_e 2}{0.0043}$

$t = 161.2$

It will take Scotland's population 161 years to double in size



7. (a) $\underline{p}\cdot(\underline{q}+\underline{r})$

$= \underline{p}\cdot\underline{q} + \underline{p}\cdot\underline{r}$

$= 4.3.\cos 30° + 4.\frac{3}{2}\cos 90°$

$= 12\frac{\sqrt{3}}{2} + 6.0$

$= 6\sqrt{3}$

$\sin 30° = \dfrac{|\underline{r}|}{|\underline{q}|}$

$\dfrac{1}{2} = \dfrac{|\underline{r}|}{3}$

$|\underline{r}| = \dfrac{3}{2}$

$\underline{r}\cdot(\underline{p}-\underline{q})$

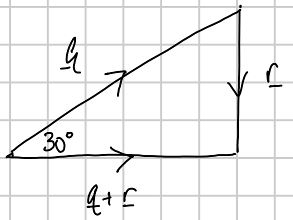
$= \underline{r}\cdot\underline{p} - \underline{r}\cdot\underline{q}$

$= \frac{3}{2}.4\cos 90° - \frac{3}{2}.3\cos 120°$

$= 6.0 - \frac{9}{2}\left(-\frac{1}{2}\right)$

$= \dfrac{9}{4}$

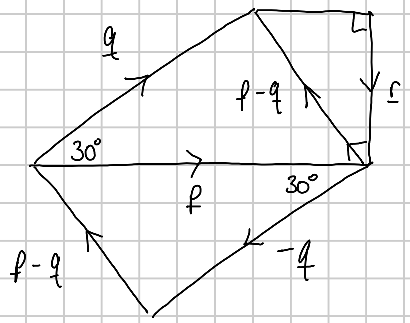
(b)

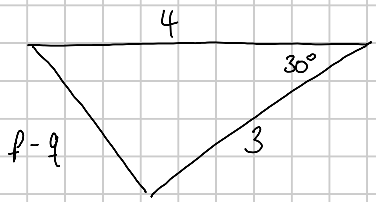


$$\cos 30^\circ = \frac{|q+r|}{3}$$

$$\frac{\sqrt{3}}{2} = \frac{|q+r|}{3}$$

$$\frac{3\sqrt{3}}{2} = |q+r|$$





(q || -q so use alternate angle)

$$(p-q)^2 = p^2 + q^2 - 2|p||q| \cos 30^\circ$$

$$(p-q)^2 = 16 + 9 - 2(4)(3)\frac{\sqrt{3}}{2}$$

$$(p-q)^2 = 25 - 12\sqrt{3}$$

$$|p-q| = \sqrt{(25 - 12\sqrt{3})}$$